In[1]:= $$\mathbf{DSolve}\left[\frac{\partial_r\left(r^2\,R'[r]\right)}{R[r]}+k^2\,r^2+L==0,\ R[r],\ r\right]$$

Out[1]= $$\left\{\left\{R[r]\to C[1]\ \mathrm{SphericalBesselJ}\left[\frac{1}{2}\left(-1+\sqrt{1-4\,L}\right),\ k\,r\right]+C[2]\ \mathrm{SphericalBesselY}\left[\frac{1}{2}\left(-1+\sqrt{1-4\,L}\right),\ k\,r\right]\right\}\right\}$$

In[2]:= $$\mathbf{DSolve}\left[\partial_t\left(\mathrm{Sin}[t]\,T'[t]\right)==\left(\frac{m^2}{\mathrm{Sin}[t]}-L\,\mathrm{Sin}[t]\right)T[t],\ T[t],\ t\right]$$

Out[2]= $$\left\{\left\{T[t]\to C[1]\ \mathrm{LegendreP}\left[\frac{1}{2}\left(-1+\sqrt{1+4\,L}\right),\ m,\ \mathrm{Cos}[t]\right]+C[2]\ \mathrm{LegendreQ}\left[\frac{1}{2}\left(-1+\sqrt{1+4\,L}\right),\ m,\ \mathrm{Cos}[t]\right]\right\}\right\}$$

$$\frac{1}{2}\left(-1+\sqrt{1+4\,L}\right)$$

In[7]:= $$L=\ell\,(\ell+1);$$

$$\mathbf{Simplify}\left[\frac{1}{2}\left(-1+\sqrt{1+4\,L}\right),\ \ell>0\right]$$

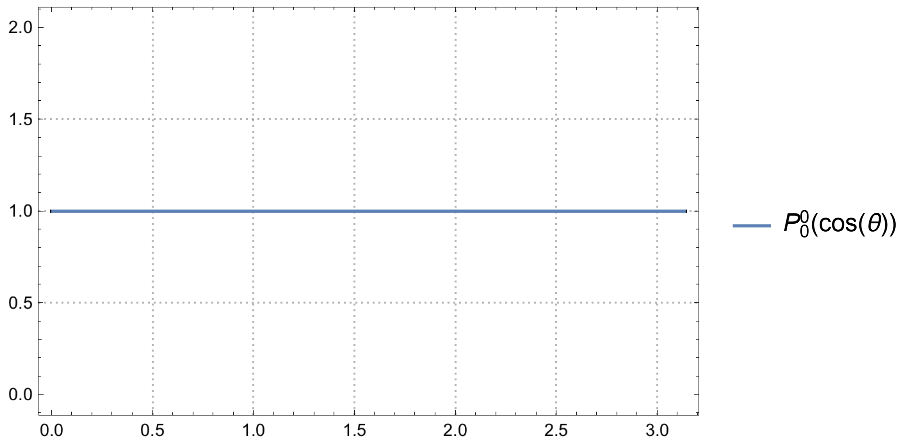
Out[8]= ℓ

Associated Legendre Polynomials

In[14]:=
```
$PlotTheme = "Detailed";

Plot[LegendreP[0, 0, Cos[θ]], {θ, 0, π}]
```

Out[15]=



In[20]:=
```
Range[-1, 1]
```

Out[20]=
```
{-1, 0, 1}
```

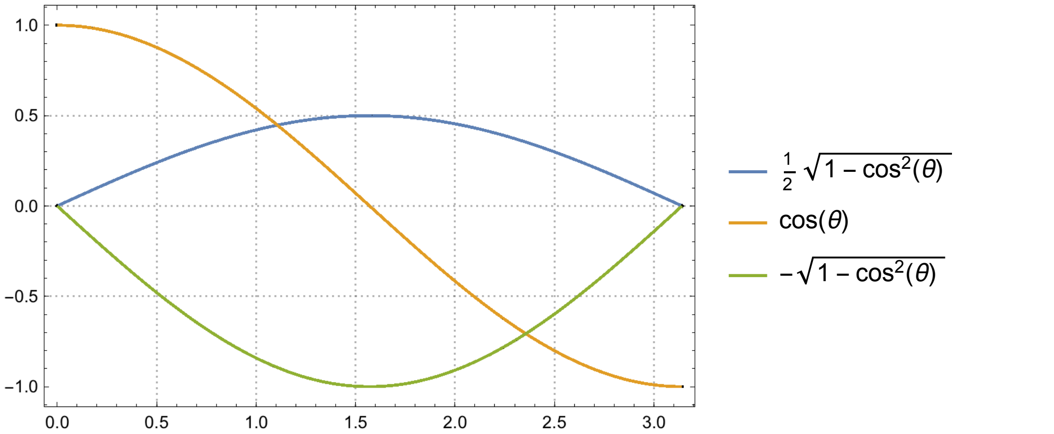
In[21]:=
```
f[#] & /@ {a, b, c}
```

Out[21]=
```
{f[a], f[b], f[c]}
```

In[22]:=
```
functions = LegendreP[1, #, Cos[θ]] & /@ Range[-1, 1]
Plot[Evaluate[functions], {θ, 0, π}]
```

Out[22]= $\left\{\frac{1}{2}\sqrt{1-\text{Cos}[\theta]^2}, \text{Cos}[\theta], -\sqrt{1-\text{Cos}[\theta]^2}\right\}$

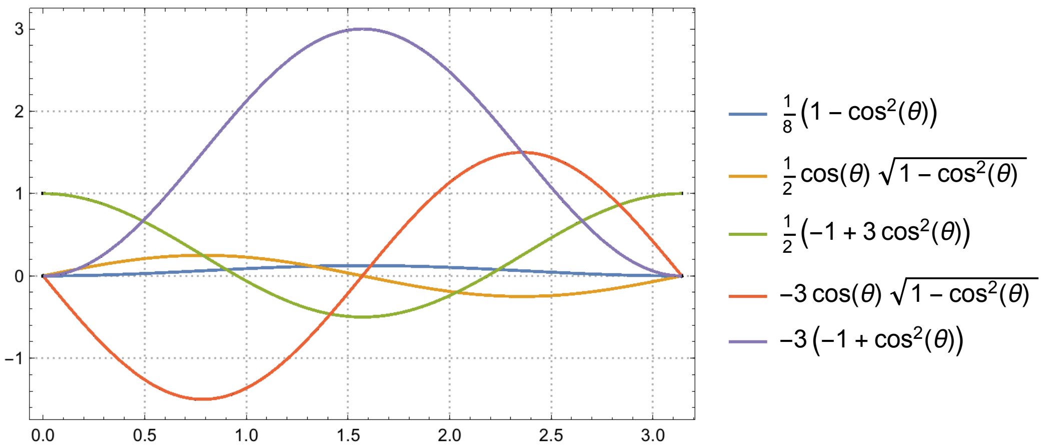
Out[23]=



In[24]:=
```
functions = LegendreP[2, #, Cos[θ]] & /@ Range[-2, 2]
Plot[Evaluate[functions], {θ, 0, π}]
```

Out[24]= $\left\{\frac{1}{8}\left(1-\text{Cos}[\theta]^2\right), \frac{1}{2}\text{Cos}[\theta]\sqrt{1-\text{Cos}[\theta]^2}, \frac{1}{2}\left(-1+3\,\text{Cos}[\theta]^2\right), -3\,\text{Cos}[\theta]\sqrt{1-\text{Cos}[\theta]^2}, -3\left(-1+\text{Cos}[\theta]^2\right)\right\}$

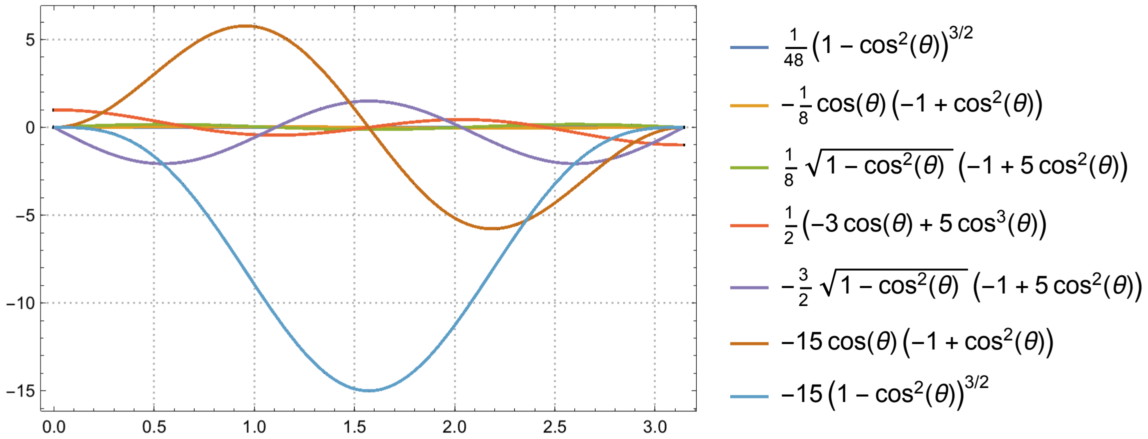
Out[25]=



In[32]:=
```
functions = LegendreP[3, #, Cos[θ]] & /@ Range[-3, 3]
Plot[Evaluate[functions], {θ, 0, π}, PlotRange → All]
```

Out[32]= $\left\{\frac{1}{48}\left(1-\text{Cos}[\theta]^2\right)^{3/2}, -\frac{1}{8}\text{Cos}[\theta]\left(-1+\text{Cos}[\theta]^2\right), \frac{1}{8}\sqrt{1-\text{Cos}[\theta]^2}\left(-1+5\,\text{Cos}[\theta]^2\right), \frac{1}{2}\left(-3\,\text{Cos}[\theta]+5\,\text{Cos}[\theta]^3\right), -\frac{3}{2}\sqrt{1-\text{Cos}[\theta]^2}\left(-1+5\,\text{Cos}[\theta]^2\right), -15\,\text{Cos}[\theta]\left(-1+\text{Cos}[\theta]^2\right), -15\left(1-\text{Cos}[\theta]^2\right)^{3/2}\right\}$

Out[33]=

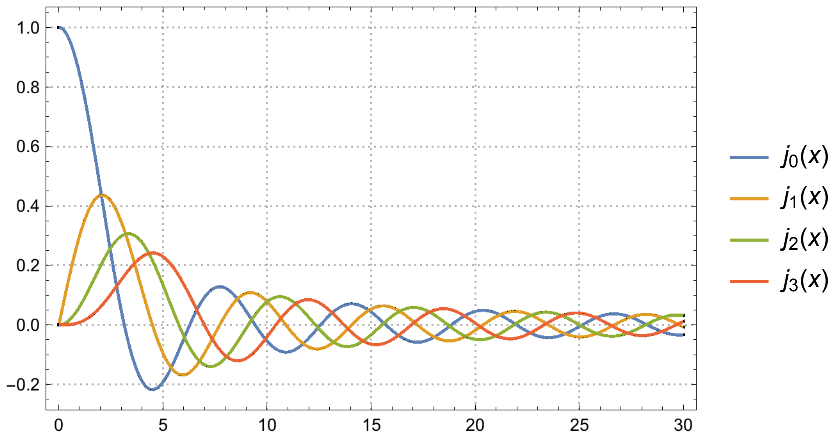


Spherical Bessel

In[82]:=

```
Plot[Evaluate[SphericalBesselJ[#, x] & /@ Range[0, 3]],
 {x, 0, 30}, PlotRange → All]
```

Out[82]=



Digamos que queiramos encontrar k tal que $j_0(k\,L) = 0$. Como encontrar k?

Devido às oscilações, não é tarefa numérica imediata, necessita de cuidados. Uma forma razoavelmente simples e direta é buscar pelas raízes numéricas de SphericalBesselJ[0,x], usando pontos finais e iniciais condizentes com o gráfico acima, ou seja,

In[63]:=
```
raiz[1] = x /. FindRoot[SphericalBesselJ[0, x] == 0, {x, 1, 0, 5}][[1]]
```

Out[63]= 3.14159

In[58]:=
```
raiz[2] = x /. FindRoot[SphericalBesselJ[0, x] == 0, {x, 5, raiz[1], 7}][[1]]
```
Out[58]= 6.28319

In[61]:=
```
raiz[3] = x /. FindRoot[SphericalBesselJ[0, x] == 0, {x, 8, raiz[2], 10}][[1]]
```
Out[61]= 9.42478

In[75]:=
```
x /. FindRoot[SphericalBesselJ[0, x] == 0, {x, 80, raiz[2], 100}][[1]]
```
Out[75]= 69.115

Assim, a n-ésima raíz de $j_0(x)$ parece ser n π (não é uma demonstração, apenas uma inspeção).

In[64]:=
```
raiz[1] / π
raiz[2] / π
raiz[3] / π
```
Out[64]= 1.

Out[65]= 2.

Out[66]= 3.

Assim, os k's possíveis são $\frac{\pi}{L}$, $\frac{2\pi}{L}$, $\frac{3\pi}{L}$, ...

Para j_1 a solução não é tão simples....

In[69]:=
```
raiz1[1] = x /. FindRoot[SphericalBesselJ[1, x] == 0, {x, 1, 0, 5}][[1]]
```
Out[69]= 0.

In[70]:=
```
raiz1[2] = x /. FindRoot[SphericalBesselJ[1, x] == 0, {x, 5, raiz1[1], 7}][[1]]
```
Out[70]= 4.49341

In[71]:=
```
raiz1[3] = x /. FindRoot[SphericalBesselJ[1, x] == 0, {x, 8, raiz1[2], 10}][[1]]
```
Out[71]= 7.72525

In[77]:=
```
raiz1[4] = x /. FindRoot[SphericalBesselJ[1, x] == 0, {x, 10, raiz1[3], 15}][[1]]
```
Out[77]= 10.9041

In[72]:=
```
raiz1[2] / π
```
Out[72]= 1.4303

In[73]:=
```
raiz1[3] / π
```
Out[73]= 2.45902

In[83]:=
```
raiz1[4] / π
```
Out[83]= 3.47089

Existe uma relação anlítica entre os zeros da função de Bessel esférica e os zeros da função de Bessel ordinária, contudo os últimos necessitam de métodos numéricos, não há forma fechada.

As raízes das funções $j_n(x)$, com n > 0 não ocorrem com periodicidade,

In[84]:=
```
raiz1[2] - raiz1[1]
raiz1[3] - raiz1[2]
raiz1[4] - raiz1[3]
```

Out[84]= 4.49341

Out[85]= 3.23184

Out[86]= 3.17887